# Arbitrage and equilibrium in unbounded exchange economies with satiation

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**Abstract**

In his seminal paper on arbitrage and competitive equilibrium in unbounded exchange economies, Werner (*Econometrica, 1987)* proved the existence of a competitive equilibrium, under a price no-arbitrage condition, without assuming either local or global nonsatiation. Werner's existence result contrasts sharply with classical existence results for bounded exchange economies which require, at minimum, global nonsatiation at rational allocations. Why do unbounded exchange economies admit existence without local or global nonsatiation? This question is the focus of our paper. *First*, we show that in unbounded exchange economies, even if some agents' preferences are satiated, the absence of arbitrage is sufficient for the existence of competitive equilibria, as long as each agent who is satiated has a nonempty set of useful net trades - that is, as long as agents' preferences satisfy weak nonsatiation. *Second*, we provide a new approach to proving existence in unbounded exchange economies. The key step in our new approach is to

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transform the original economy to an economy satisfying global nonsatiation such that all equilibria of the transformed economy are equilibria of the original economy. What our approach makes clear is that it is precisely the condition of weak nonsatiation - a condition considerably weaker than local or global nonsatiation - that makes possible this transformation.

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## 1. Introduction

Since the pioneering contributions of Grandmont ((1970), (1972), (1977)), Green (1973), and Hart (1974), the relationship between arbitrage and equilibrium in asset exchange economies allowing short sales has been the subject of much investigation.[1] When unlimited short sales are allowed, agents' choice sets are unbounded from below. As a consequence, asset prices at which agents can exhaust all gains from trade via mutually compatible net trades bounded in size may fail to exist. By assuming that markets admit "no arbitrage", the economy can be *bounded endogenously* - but this is not enough for existence. In addition to no-arbitrage conditions, two other conditions are frequently required: (i) *uniformity* of arbitrage opportunities, and (ii) *nonsatiation*. Werner, in his seminal 1987 paper on arbitrage and competitive equilibrium, assumes uniformity of arbitrage opportunities and establishes the existence of a competitive equilibrium using a no-arbitrage condition on prices. An especially intriguing aspect of Werner's existence result is that it does not require local or global nonsatiation (see Werner (1987), Theorems 1). This contrasts sharply with classical existence results for bounded exchange economies which require, at minimum, that agents' preferences be globally nonsatiated at rational allocations (e.g., see Debreu (1959), Gale and Mas-Colell (1975), and Bergstrom (1976)). Why do unbounded exchange economies admit existence without local or global nonsatiation? This question is the focus of our paper.

Our starting point is Werner's notion of useful net trades. Stated informally, a useful net trade is a net trade that, for some endowments, represents a potential arbitrage. Our main contribution is to show that, in general, in unbounded exchange economies (for example, asset exchange economies allowing short sales), even if some agents' preferences are satiated, the absence of market arbitrage is sufficient for the existence of competitive equilibria, *as long as each agent who*

[1] See also, for example, Milne (1980), Hammond (1983), Page (1987), Nielsen (1989), Page and Wooders (1996), Kim (1998), Dana, Le Van, Magnien (1999), Page, Wooders, and Monteiro (2000), and Allouch and Florenzano (2004).

*is satiated has a nonempty set of useful net trades* - that is, as long as agents' preferences satisfy *weak nonsatiation*.

Our second contribution is to provide a new approach to proving existence in unbounded exchange economies. In addition to being a technical innovation, our new approach makes clear the critical role played by unboundedness and weak nonsatiation in establishing existence in unbounded exchange economies where neither local nor global nonsatiation is satisfied. The key step in our approach is a transformation of the original economy to a new economy satisfying global nonsatiation and having the property that all equilibria of the transformed economy are equilibria of the original economy. Existence for the transformed economy is then deduced using classical methods. It is precisely the condition of weak nonsatiation - a condition considerably weaker than local or global nonsatiation - that makes possible the transformation of the original economy to an equivalent economy satisfying global nonsatiation - even if the original economy fails to satisfy either local or global nonsatiation.

In the literature there is a well-established tradition of considering an artificial economy to prove results for the original economy. In their classic paper on abstract exchange economies, Gale and Mas-Colell (1975) establish existence by transforming an exchange economy satisfying global nonsatiation to an exchange economy satisfying local nonsatiation. However, if global nonsatiation fails, then the Gale/Mas-Colell transformation cannot be applied. Here, we establish existence by transforming an exchange economy satisfying weak nonsatiation (in which global nonsatiation may fail) to an exchange economy satisfying global nonsatiation. Thus, while our transformation is similar in motivation to the Gale/Mas-Colell transformation, it goes beyond the Gale/Mas-Colell transformation by addressing the problem of global satiation. We also extend Werner's model of an unbounded exchange economy in two ways. First, we weaken Werner's uniformity of arbitrage condition by assuming only uniformity of useless net trades (see Werner (1987), Assumption A3). We refer to our uniformity condition as weak uniformity. Second, in our model we require only that agents' utility functions be upper semicontinuous, rather than continuous as in Werner (1987).

We shall proceed as follows: In Section 2, we present the basic ingredients of our model, including the notions of arbitrage, useful and useless net trades, weak uniformity, and weak nonsatiation. We also discuss the weak no-market-arbitrage condition of Hart (1974) and the price no-arbitrage condition of Werner (1987), and we extend Werner's price no-arbitrage condition to allow for weak nonsatiation. In Section 3, we present our main result which states that in an unbounded exchange economy (for example, in an asset exchange economy allowing short sales), if weak uniformity and weak nonsatiation hold, then the extended price no-arbitrage condition is sufficient to guarantee the existence of a quasi-equilibrium - and therefore is sufficient to guarantee the existence of a competitive equilibrium under the usual relative interiority conditions on endowments. In Section 4, we present an example which shows that Werner's nonsatiation is different from global nonsatiation. Finally, in Section 5 we present the proof of our main result, Theorem 1. In the proof of Theorem 1, we present our new approach.

## 2. The Model

We consider an economy $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ with $m$ agents and $l$ goods. Agent $i$ has consumption set $X_i \subset \mathbb{R}^l$, utility function $u_i(\cdot)$, and endowment $e_i$. Agent $i's$ preferred set at $x_i \in X_i$ is

$$P_i(x_i) = \{x \in X_i \mid \ u_i(x) > u_i(x_i)\},$$

while the weakly preferred set at $x_i$ is

$$\widehat{P}_i(x_i) = \{x \in X_i \mid \ u_i(x) \geq u_i(x_i)\}.$$

The set of *individually rational allocations* is given by

$$\mathcal{A} = \{(x_i) \in \prod_{i=1}^m X_i \mid \sum_{i=1}^m x_i = \sum_{i=1}^m e_i \text{ and } \ x_i \in \widehat{P}_i(e_i), \forall i\}.$$

We shall denote by $\mathcal{A}_i$ the projection of $\mathcal{A}$ onto $X_i$.

**Definition 1.** *(a) A rational allocation $x^* \in \mathcal{A}$ together with a* nonzero *vector of prices $p^* \in \mathbb{R}^l$ is an equilibrium for the economy $\mathcal{E}$*

*(i) if for each agent $i$ and $x \in X_i$, $u_i(x) > u_i(x_i^*)$ implies $p^* \cdot x > p^* \cdot e_i$, and*

*(ii) if for each agent $i$, $p^* \cdot x_i^* = p^* \cdot e_i$.*

*(b) A rational allocation $x^* \in \mathcal{A}$ and a* nonzero *price vector $p^* \in \mathbb{R}^l$ is a quasi-equilibrium*

*(i) if for each agent $i$ and $x \in X_i$, $u_i(x) > u_i(x_i^*)$ implies $p^* \cdot x \geq p^* \cdot e_i$, and*

*(ii) if for each agent $i$, $p^* \cdot x_i^* = p^* \cdot e_i$.*

Given $(x^*, p^*)$ a quasi-equilibrium, it is well-known that if for each agent $i$, (a) $p^* \cdot x < p^* \cdot e_i$ for some $x \in X_i$ and (b) $P_i(x_i^*)$ is relatively open in $X_i$, then $(x^*, p^*)$ is an equilibrium. Conditions (a) and (b) will be satisfied if, for example, for each agent $i$, $e_i \in \mathrm{int} X_i$, and $u_i$ is continuous on $X_i$. Using irreducibility assumptions, one can also show that a quasi-equilibrium is an equilibrium.

We now introduce our first two assumptions: for agents $i = 1, 2, \ldots, m$,

[A.1] $X_i$ is closed and convex with $e_i \in X_i$,

[A.2] $u_i$ is upper semicontinuous and quasi-concave.

Under these two assumptions, the weak preferred set $\widehat{P}_i(x_i)$ is convex and closed for $x_i \in X_i$.

### 2.1. Arbitrage, Uniformity, and Nonsatiation

#### 2.1.1. Arbitrage

We define the $i^{th}$ *agent's arbitrage cone at* $x_i \in X_i$ as the closed convex cone containing the origin given by

$$O^+\widehat{P}_i(x_i) = \{y_i \in \mathbb{R}^l \mid \forall x_i' \in \widehat{P}_i(x_i) \text{ and } \lambda \geq 0,\ x_i' + \lambda y_i \in \widehat{P}_i(x_i)\}.$$

Thus, if $y_i \in O^+\widehat{P}_i(x_i)$, then for all $\lambda \geq 0$ and all $x_i' \in \widehat{P}_i(x_i)$, $x_i' + \lambda y_i \in X_i$ and $u_i(x_i' + \lambda y_i) \geq u_i(x_i)$. The agent's arbitrage cone at $x_i$, then, is the recession cone corresponding to the weakly preferred set $\widehat{P}_i(x_i)$ (see Rockafellar (1970), Section 8).[2] If the agent's utility function, $u_i(\cdot)$, is concave, then for any $x_i \in X_i$, $x_i' \in \widehat{P}_i(x_i)$, and $y_i \in O^+\widehat{P}_i(x_i)$, $u_i(x_i' + \lambda y_i)$ is nondecreasing in $\lambda \geq 0$. Thus, starting at any $x_i' \in \widehat{P}_i(x_i)$, trading in the $y_i$ direction on any scale is utility nondecreasing. Moreover, if $u_i(\cdot)$, is strictly concave, then for any $x_i \in X_i$, $x_i' \in \widehat{P}_i(x_i)$, and nonzero $y_i \in O^+\widehat{P}_i(x_i)$, $u_i(x_i' + \lambda y_i)$ is increasing in $\lambda \geq 0$. Thus for $u_i(\cdot)$ is strictly concave, starting at any $x_i' \in \widehat{P}_i(x_i)$, trading in the $y_i$ direction ($y_i \neq 0$) on any scale $\lambda \geq 0$ is utility increasing.

[2] Equivalently, $y_i \in O^+\widehat{P}_i(x_i)$ if and only if $y_i$ is a cluster point of some sequence $\{\lambda^k x_i^k\}_k$ where the sequence of positive numbers $\{\lambda^k\}_k$ is such that $\lambda^k \downarrow 0$, and where for all $k$, $x_i^k \in \widehat{P}_i(x_i)$; (see Rockafellar (1970), Theorem 8.2).

### 2.1.2. Uniformity

A set closely related to the $i^{th}$ agent's arbitrage cone is the *lineality space*, $L_i(x_i)$, of $\widehat{P}_i(x_i)$ given by

$$L_i(x_i) = \{y_i \in \mathbb{R}^l \mid \ \forall x_i' \in \widehat{P}_i(x_i) \text{ and } \forall \lambda \in \mathbb{R},\ x_i' + \lambda y_i \in \widehat{P}_i(x_i)\}.$$

The set $L_i(x_i)$ consists of the zero vector and all the nonzero vectors $y_i$ such that for each $x_i'$ weakly preferred to $x_i$ (i.e., $x_i' \in \widehat{P}_i(x_i)$), any vector $z_i$ on the line through $x_i'$ in the direction $y_i$, $z_i = x_i' + \lambda y_i$, is also weakly preferred to $x_i$ (i.e., $z_i = x_i' + \lambda y_i \in \widehat{P}_i(x_i)$). The set $L_i(x_i)$ is a closed subspace of $\mathbb{R}^l$, and is the largest subspace contained in the arbitrage cone $O^+\widehat{P}_i(x_i)$ (see Rockafellar (1970)).

If for all agents, the lineality space $L_i(x_i)$ is the same for all $x_i \in \widehat{P}_i(e_i)$, then we say that the economy satisfies *weak uniformity*. We formalize this notion of uniformity in the following assumption:

$$\text{[A.3][Weak Uniformity] for all agents } i$$
$$L_i(x_i) = L_i(e_i) \text{ for all } x_i \in \widehat{P}_i(e_i).$$

Under weak uniformity, we have for all $x_i \in \widehat{P}_i(e_i)$ and all $y_i \in L_i(e_i)$,

$$u_i(x_i + y_i) \leq u_i(x_i + y_i - y_i) \leq u_i(x_i + y_i).$$

Thus, for all $x_i \in \widehat{P}_i(e_i)$ and all $y_i \in L_i(e_i)$,

$$u_i(x_i + y_i) = u_i(x_i).$$

Following the terminology of Werner (1987), we refer to arbitrage opportunities $y_i \in O^+\widehat{P}_i(x_i)$ such that

$$u_i(x_i + \lambda y_i) = u_i(x_i) \text{ for all } \lambda \in (-\infty, \infty)$$

as *useless* at $x_i$. Thus, under weak uniformity, the $i^{th}$ agent's lineality space at his endowment, $L_i(e_i)$, is equal to the set of all net trades that are useless. Moreover, under weak uniformity the set of *useful* net trades at $x_i$ is given by

$$O^+\widehat{P}_i(x_i)\backslash L_i(x_i) = O^+\widehat{P}_i(x_i)\backslash L_i(e_i).$$

Werner (1987) makes a uniformity assumption stronger than our assumption of uniformity of useless net trades (i.e., stronger than our assumption of weak uniformity, [A.3]). In particular, Werner assumes that *all* arbitrage opportunities are uniform. Stated formally,

$$\text{[Uniformity] for all agents } i$$
$$O^+\widehat{P}_i(x_i) = O^+\widehat{P}_i(e_i) \text{ for all } x_i \in \widehat{P}_i(e_i).$$

If agents have concave utility functions, then Werner's uniformity assumption, and therefore weak uniformity, is satisfied automatically.

For notational simplicity, we will denote each agent's arbitrage cone and lineality space at endowments in a special way. In particular, we will let

$$R_i := O^+\widehat{P}_i(e_i), \text{ and } L_i := L(e_i).$$

### 2.1.3. Nonsatiation

We begin by recalling the classical notions of global and local nonsatiation:

$$\text{[GlobalNonsatiation] for all agents } i,$$
$$P_i(x_i) \neq \emptyset \text{ for all } x_i \in \mathcal{A}_i;$$

$$\text{[LocalNonsatiation] for all agents } i,$$
$$P_i(x_i) \neq \emptyset \text{ and } \mathrm{cl}P_i(x_i) = \widehat{P}_i(x_i) \text{ for all } x_i \in \mathcal{A}_i.$$

Here, cl denotes closure. Werner assumes uniformity and then, rather than assume global or local nonsatiation, assumes that

$$\text{[Werner Nonsatiation] for all agents } i$$
$$R_i \backslash L_i \neq \varnothing.$$

This assumption is weaker than the classical assumptions. We will weaken Werner's nonsatiation assumption as follows:

$$\text{[A.4][Weak Nonsatiation] for all agents } i$$
$$\forall x_i \in \mathcal{A}_i, \text{ if } P_i(x_i) = \emptyset, \text{ then } O^+\widehat{P}_i(x_i) \setminus L_i(x_i) \neq \emptyset.$$

Note that weak nonsatiation holds if global nonsatiation, local nonsatiation, or Werner nonsatiation holds. Also, note that under weak nonsatiation if $x_i \in \mathcal{A}_i$ is a satiation point for agent $i$, then, as in Werner, there is a *useful* net trade vector $y_i$ such that $u_i(x_i + \lambda y_i) = u_i(x_i)$ for all $\lambda \geq 0$. Thus, if there are satiation points, then the set of satiation points must be *unbounded*.

## 2.2. The No-Arbitrage Conditions of Hart and Werner

Hart's (1974) no-arbitrage condition is a condition on net trades. In particular, Hart's condition requires that all mutually compatible arbitrage opportunities be *useless*.[3] We shall refer to Hart's condition as the *weak no-market-arbitrage* condition (WNMA). We have the following definition:

[3] Hart's condition is stated within the context of an asset exchange economy model where uncertainty concerning asset returns is specified via a joint probability distribution function. Page (1987) shows that in an asset exchange economy, if there are no perfectly correlated assets, then Hart's condition and Page's (1987) no-unbounded-arbitrage condition are equivalent.

**Definition 2.** *The economy $\mathcal{E}$ satisfies the WNMA condition if*

$$\begin{array}{c}\sum_{i=1}^{m} y_i = 0 \text{ and } y_i \in R_i \text{ for all } i, \text{ then} \\ y_i \in L_i \text{ for all } i.\end{array}$$

Werner's (1987) no-arbitrage condition is a condition on prices. In particular, Werner's condition requires that there be a nonempty set of prices such that each price in this set assigns a strictly positive value to any vector of *useful* net trades belonging to any agent.[4] We shall refer to Werner's condition as the *price no-arbitrage* condition (PNA). We have the following definition:

**Definition 3.** *In an economy $\mathcal{E}$ satisfying [Werner Nonsatiation], Werner's PNA condition is satisfied if*

$$\bigcap_{i=1}^{m} S_i^W \neq \emptyset,$$

*where*

$$S_i^W = \{p \in \mathbb{R}^\ell \mid \ p \cdot y > 0, \forall y \in R_i \setminus L_i\}$$

*is Werner's cone of no-arbitrage prices.*

Allouch, Le Van, and Page (2002) extend Werner's condition to allow for the possibility that for some agent the set of useful net trades is empty - that is, to allow for the possibility that for some agent, $R_i \setminus L_i = \emptyset$. More importantly, they prove, under very mild conditions, that their extended version of Werner's condition is equivalent to Hart's condition.

Here we follow their extension of Werner's cone of *no-arbitrage prices*:

**Definition 4.** *For each agent $i$, define*

$$S_i = \begin{cases} S_i^W & \text{if } R_i \setminus L_i \neq \emptyset, \\ L_i^\perp & \text{if } R_i \setminus L_i = \emptyset, \end{cases}$$

*where* $L_i^\perp := \left\{p \in \mathbb{R}^l : p \cdot x = 0 \text{ for all } x \in L_i\right\}$.

[4] Translating Werner's condition to an asset exchange economy, it is easy to show that if there are no perfectly correlated assets and if agents are sufficiently risk averse, then Werner's condition is equivalent to Hammond's overlapping expectation condition. Page (1987) shows that in an asset exchange economy if there are no perfectly correlated assets and if agents are sufficiently risk averse, then Hammond's overlapping expectations condition and Page's no-unbounded-arbitrage condition are equivalent. Thus, in an asset exchange economy with no perfectly correlated assets populated by sufficiently risk averse agents, the conditions of Hart (1974), Werner (1987), Hammond (1983), and Page (1987) are all equivalent.

Given this expanded definition of the no-arbitrage-price cone, the *extended price no-arbitrage* condition (EPNA) is defined as follows:

**Definition 5.** *The economy $\mathcal{E}$ satisfies the EPNA condition if*

$$\bigcap_{i=1}^{m} S_i \neq \emptyset.$$

Note that if the economy $\mathcal{E}$ satisfies Werner's nonsatiation condition, i.e., $R_i \setminus L_i \neq \emptyset$, $\forall i$, then the EPNA condition given in Definition 5 above reduces to Werner's original condition PNA given in Definition 3.

We will use the following result from Allouch, Le Van, and Page (2002).

**Proposition 1.** *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying [A.1]-[A.2]. The following statements are equivalent:*

1. *$\mathcal{E}$ satisfies* WNMA.
2. *$\mathcal{E}$ satisfies EPNA.*

## 3. The Existence of Equilibrium

Our next result extends Werner's (1987) main result on arbitrage and the existence of equilibrium in two ways:

(1) Werner assumes uniformity of arbitrage opportunities. Here, we assume only weak uniformity [A.3] (i.e., uniformity of agents' lineality space).

(2) Werner assumes that for each agent $i$, $O^+\widehat{P}_i(x_i) \setminus L_i(x_i) \neq \emptyset$, $\forall x_i \in X_i$. Here, we weaken Werner's nonsatiation assumption to allow $O^+\widehat{P}_i(x_i) = L_i(x_i)$ for some agents $i$ and some $x_i \in \mathcal{A}_i$. But in this case we require that $P_i(x_i) \neq \emptyset$. In particular, we require only weak nonsatiation [A.4].

**Theorem 1.** *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying [A.1]-[A.2], weak uniformity [A.3], and weak nonsatiation* [A.4]. *If $\mathcal{E}$ satisfies Hart's condition, WNMA, or equivalently, if $\mathcal{E}$ satisfies the extended Werner condition, EPNA, then $\mathcal{E}$ has a quasi-equilibrium.*

*Moreover, if $(x_1^*, \ldots, x_m^*, p^*)$ is a quasi-equilibrium of $\mathcal{E}$ such that for each agent $i$,*

1. *$\inf_{x \in X_i} \langle x, p \rangle < \langle \omega_i, p \rangle$, and*

2. *$P_i(x_i^*)$ is relatively open in $X_i$,*

*then $(x_1^*, \dots, x_m^*, p^*)$ is an equilibrium.*

**P roof**: See Section 5.

In addition to extending Werner (1987), we also introduce a new method for proving existence in exchange economies with short selling. In particular, we prove existence by first transforming the economy $\mathcal{E}$ to an economy $\mathcal{E}'$ satisfying global nonsatiation and having the property that any equilibrium of $\mathcal{E}'$ is an equilibrium of $\mathcal{E}$. We accomplish this via a modification of agents' utility functions. Our assumption of weak nonsatiation is crucial - it allows us to modify agents' utility functions in precisely the right way. We then prove existence for the modified economy $\mathcal{E}'$ using the excess demand approach via the Gale-Nikaido-Debreu Lemma.

## 4. Example

Weak nonsatiation [A.4] plays a critical role in our proof of existence. In this section, we present an example which shows that Werner's nonsatiation differs from the standard global nonsatiation and that our weak nonsatiation assumption is the weakest possible. In our example, there are two consumers and two goods. Consumer 1, satisfies Werner's nonsatiation, but fails to satisfy global nonsatiation. Consumer 2, fails to satisfy Werner nonsatiation, but satisfies global nonsatiation. However, the economy does satisfy weak nonsatiation, as well as all the assumptions of our Theorem 1 - and there exists a quasi-equilibrium. In the example, as in Werner (1987), there is uniformity of arbitrage opportunities.

Consider the economy with two consumers and two commodities. Consumer 1 has consumption set $X_1 = R^2$, and endowment $e_1 = (3, 1)$. Consumer 1's utility function is given by

$$u_1(x_1, y_1) = \begin{cases} \min\{x_1, y_1\}, & \text{if either } x_1 \in [0,1] \text{ or } y_1 \in [0,1], \\ 1 & \text{otherwise.} \end{cases}$$

Consumer 2 has consumption set $X_2 = [0, 6]^2$, and endowment $e_2 = (1, 3)$. Consumer 2's utility function is given by $u_2(x_2, y_2) = x_2 y_2$.

We have $u_1(e_1) = 1$ and $u_2(e_2) = 3$, and the set of individually rational feasible allocations is:

$$\mathcal{A} = \{\zeta_1, \zeta_2 \in X_1 \times X_2 \mid \zeta_1 + \zeta_2 = (4, 4), \text{ and } u_1(\zeta_1) \geq 1, u_2(\zeta_2) \geq 3\}.$$

For consumer 1, Werner's nonsatiation is satisfied, whereas global nonsatiation fails. Indeed, $(2,2)$ is in the set of attainable consumption plan for consumer 1. But $(2,2)$ is a satiation point. Hence, global nonsatiation is not satisfied for consumer 1. However, Werner's nonsatiation condition is satisfied since the portfolio $(1,1)$ is a useful net trade for consumer 1. For consumer 2, Werner's nonsatiation fails while global nonsatiation is satisfied. Indeed, there is no useful portfolio for consumer 2, since his arbitrage cone is $R_2 = \{0\}$. On the other hand, $(5,5)$ is not an individually feasible bundle, and it delivers strictly greater utility than all individually feasible bundles in $\mathcal{A}_2 \subset [0,4]^2$.

It is easy to see that Hart's condition (WNMA) is satisfied, and for both consumers our weak nonsatiation assumption is satisfied. Also, it is easy to check that

$$(x_1^*, x_2^*, p^*) = (((2,2),(2,2),(1,1))$$

is a quasi-equilibrium.

## 5. Proof of Existence

### 5.1. Modifying the economy

Our method of proving existence is new. Our starting point is an exchange economy $\mathcal{E}$ satisfying assumptions [A.1]-[A.2] and weak nonsatiation [A.4]. To deal with the problem of satiation, we construct a new economy $\mathcal{E}'$ in which agents' utility functions have been modified. In the new economy $\mathcal{E}'$ agents' preferences are such that no agent is satiated at a rational allocation. Below, we establish that if the economy $\mathcal{E}$ satisfies assumptions [A.1]-[A.2] and weak nonsatiation [A.4], then the modified economy $\mathcal{E}'$ satisfies assumptions [A.1]-[A.2], and global nonsatiation. Moreover, we show that if $\mathcal{E}$ satisfies Hart's condition, then the modified economy $\mathcal{E}'$ also satisfies Hart's condition. Finally, we show that a quasi-equilibrium for the modified economy $\mathcal{E}'$ is also a quasi-equilibrium for the original economy $\mathcal{E}$.

Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying [A.1]-[A.2] and weak nonsatiation [A.4]. We begin by modifying agents' utility functions. Suppose that for some agent $i$ there exists a satiation point $x_i^* \in \mathcal{A}_i$, that is,

$$u_i(x_i^*) = \sup_{x_i \in X_i} u_i(x_i).$$

It follows from weak nonsatiation [A.4] that there exists

$$r_i \in O^+ \widehat{P}_i(x_i^*) \setminus L_i(x_i^*).$$

Using $r_i$ we define the function

$$\rho_i(\cdot) : \widehat{P}_i(x_i^*) \to \mathbb{R}_+$$

as follows:

$$\rho_i(x_i) = \sup\{\beta \in \mathbb{R}_+ \mid\ (x_i - \beta r_i) \in \widehat{P}_i(x_i^*)\}.$$

Now using the function $\rho_i(\cdot)$, we can define a new utility function, $v_i(\cdot)$, for agent $i$:

$$v_i(x_i) = \begin{cases} u_i(x_i) + \rho_i(x_i), & \text{if } x_i \text{ is a satiation point,} \\ u_i(x_i), & \text{otherwise.} \end{cases}$$

**Claim 1.** *The function $\rho_i$ is well-defined. Moreover, for all $x_i \in \widehat{P}_i(x_i^*)$ we have $(x_i - \rho_i(x_i)r_i) \in \widehat{P}_i(x_i^*)$.*

*Proof of Claim 1.* Let

$$W = \{\beta \in \mathbb{R}_+ \mid\ (x_i - \beta r_i) \in \widehat{P}_i(x_i^*)\}.$$

We first notice that $0 \in W$. Thus, $\emptyset \neq W \subset \mathbb{R}_+$. We claim that $W$ is bounded. Suppose the contrary. Then $-r_i \in O^+\widehat{P}_i(x_i^*)$ and therefore $r_i \in L_i(x_i^*)$, which contradicts $r_i \in \widehat{P}_i(x_i^*) \setminus L_i(x_i^*)$. Finally, we have $(x_i - \rho_i(x_i)r_i) \in \widehat{P}_i(x_i^*)$ since $\widehat{P}_i(x_i^*)$ is closed.□

**Claim 2.** *Let $\lambda \geq 0$. Then*

$$\{x \in \widehat{P}_i(x_i^*) \mid\ \rho(x) \geq \lambda\} = \{\lambda r_i\} + \widehat{P}_i(x_i^*).$$

*Proof of Claim 2.* First it is obvious that

$$\{\lambda r_i\} + (\widehat{P}_i(x_i^*)) \subset \{x \in \widehat{P}_i(x_i^*) \mid\ \rho_i(x) \geq \lambda\}.$$

Furthermore, let $x_i \in \{x \in \widehat{P}_i(x_i^*) \mid\ \rho_i(x) \geq \lambda\}$. Then, $(x_i - \rho(x_i)r_i) \in \widehat{P}_i(x_i^*)$ and therefore $x_i \in \{\lambda r_i\} + \widehat{P}_i(x_i^*)$, since $\widehat{P}_i(x_i^*)$ is convex.□

**Claim 3.** *We have* $\sup_{x_i \in \widehat{P}_i(x_i^*)} \rho_i(x_i) = +\infty$.

*Proof of Claim 3.* It is obvious that $(x_i + \lambda r_i) \in \widehat{P}_i(x_i^*)$, for all $\lambda \geq 0$, since $r_i \in O^+\widehat{P}_i(x_i^*)$. Moreover, $\rho_i(x_i + \lambda r_i) \geq \lambda$. Then, $\sup_{x_i \in \widehat{P}_i(x_i^*)} \rho_i(x_i) = +\infty$.□

Consider the level set $E_\lambda = \{x \in X_i \mid\ v_i(x) \geq \lambda\}$, for every $\lambda \in \mathbb{R}$.

**Claim 4.** *The function $v_i$ is upper semicontinuous and quasi-concave. Moreover, for all $x_i \in \widehat{P}_i(e_i)$*

$$O^+E_{v_i(x_i)} = \begin{cases} O^+\widehat{P}_i(x_i^*), & \text{if } x_i \text{ is a satiation point,} \\ O^+\widehat{P}_i(x_i), & \text{otherwise.} \end{cases}$$

*Proof of Claim 4.* The function $v_i$ is upper semicontinuous and quasi-concave if and only if $E_\lambda$ is closed and convex for all $\lambda \in \mathbb{R}$.

*first case.* Suppose $\lambda \leq u_i(x_i^*)$. Then, $E_\lambda = \{x \in X_i \mid \; u_i(x) \geq \lambda\}$. Thus, $E_\lambda$ is closed and convex, since $u_i$ is upper semicontinuous and quasi-concave.

*second case.* Suppose $\lambda > u_i(x_i^*)$. Then

$$\begin{aligned} E_\lambda &= \{x \in X_i \mid \; v_i(x) \geq \lambda\} \\ &= \{x \in \widehat{P}_i(x_i^*) \mid \; \rho_i(x) \geq (\lambda - u_i(x_i^*))\} \\ &= \{(\lambda - u_i(x_i^*))r_i\} + \widehat{P}_i(x_i^*). \end{aligned}$$

Thus, $E_\lambda$ is convex and closed.□

Now, we consider the modified economy $\mathcal{E}' = (X_i, v_i, e_i)_{i=1,\dots,m}$. Let

$$\mathcal{A}' = \{(x_i) \in \prod_{i=1}^m X_i \mid \sum_{i=1}^m x_i = \sum_{i=1}^m e_i \text{ and } \; v_i(x_i) \geq v_i(e_i), \forall i\},$$

be the set of rational allocations of $\mathcal{E}'$.

**Claim 5.** *If in addition to satisfying assumptions [A.1]-[A.2], and weak nonsatiation* [A.4], *$\mathcal{E}$ also satisfies weak uniformity [A.3], then the following statement is true:*

*If the original economy $\mathcal{E}$ satisfies Hart's condition (WNMA), then the modified economy $\mathcal{E}'$ also satisfies Hart's condition.*

*Proof of Claim 5.* It follows from Claim 4 that for all $x_i \in E_{v_i(e_i)}$ we have

$$L_i \subset O^+E_{v_i(x_i)} \subset O^+E_{v_i(e_i)} \subset R_i.$$

Since, $L_i$ is the maximal subspace in $R_i$, one gets $v_i$ has uniform lineality space equal to $L_i$. Furthermore, $\sum_{i=1}^m y_i = 0$ with $\forall i$, $y_i \in O^+E_{v_i(e_i)}$ implies that $\sum_{i=1}^m y_i = 0$ with $\forall i$, $y_i \in R_i$. Since $\mathcal{E}$ satisfies the WNMA condition, $y_i \in L_i$, $\forall i$. Therefore, $\mathcal{E}'$ also satisfies the WNMA condition .□

**Claim 6.** *We have:*

*(i) The modified economy $\mathcal{E}'$ satisfies Global Nonsatiation.*

*(ii) If $(x^*, p^*)$ is a quasi-equilibrium of $\mathcal{E}'$, then $(x^*, p^*)$ is a quasi-equilibrium of $\mathcal{E}$.*

*Proof of Claim 5.6.* $(i)$ It follows from Claim 3.

$(ii)$ It is clear that $x^* \in \mathcal{A}' \subset \mathcal{A}$. Moreover, let $x_i \in X_i$ be such that $u_i(x_i) > u_i(x_i^*)$. Then, $x_i^*$ is not a satiation point and therefore $v_i(x_i^*) = u_i(x_i^*)$. Since $v_i(x_i) \geq u_i(x_i)$, it follows that $v_i(x_i) > v_i(x_i^*)$. Since $(x^*, p^*)$ is a quasi-equilibrium of $\mathcal{E}'$, we can conclude that $p^* \cdot x_i \geq p^* \cdot e_i$. Thus, $(x^*, p^*)$ is a quasi-equilibrium of $\mathcal{E}$.□

### 5.2. Proof of Theorem 1

First, it follows from Claim 5 that $\mathcal{E}'$ also satisfies the WNMA. From Claim 5.6 it is sufficient to show that $\mathcal{E}'$ has a quasi-equilibrium.

We consider a sequence of truncated economies with consumption sets

$$X_i^n = \widehat{P}_i(e_i) \cap L_i^{\perp} \cap \mathrm{cl}B(0,n),$$

where $B(0,n)$ is the open ball of radius $n$ centered at 0. We choose $n$ large enough so that $e_i \in B(0,n)$ for each $i$.
Let $D = \cap L_i^{\perp}$ and $\Pi$ is the unit sphere of $\mathbb{R}^{l+1}$.

For $(p,q) \in (D \times \mathbb{R}_+) \cap \Pi$, we consider

$$\varphi_i^n(p,q) = \{x_i \in X_i^n \mid\ p \cdot x_i \leq p \cdot e_i + q\},$$

and

$$\zeta_i^n(p,q) = \{x_i \in \varphi_i^n(p,q) \mid\ y \in \widetilde{P}_i^n(x_i) \Rightarrow p \cdot y \geq p \cdot e_i + q\},$$

where

$$\widetilde{P}_i^n(x_i) = \{(1-\lambda)x_i + \lambda z_i \mid\ 0 < \lambda \leq 1, v_i(x_i) < v_i(z_i) \text{ and } z_i \in X_i^n\}.$$

We have the following result:

**Lemma 1.** *For $n$ large enough, $\zeta_i^n$ is upper semicontinuous nonempty, compact and convex valued, for every $i$.*

**P roof**: First we show that $\zeta_i^n(p,q)$ is nonempty for n large enough. For $n$ large enough, $e_i^{\perp} \in \varphi_i^n(p,q)$. Let $\widehat{x}_i$ be a maximizer of $v_i$ on $\varphi_i^n(p,q)$. If $\widetilde{P}_i^n(\widehat{x}_i) = \emptyset$, we end the proof, since $\widehat{x}_i \in \zeta_i^n(p,q)$. If not, let $z_i \in X_i^n$, such that $v_i(z_i) > v_i(\widehat{x}_i)$.

By the very definition of $\widehat{x}_i$, we have $p \cdot z_i > p \cdot e_i + q$. Let $t_i$, contained in the segment $[\widehat{x}_i, z_i]$, be such that

$$p \cdot t_i = p \cdot e_i + q.$$

By quasi-concavity of the utility function, $v_i(t_i) \geq v_i(\widehat{x}_i)$. By the definition of $\widehat{x}_i, v_i(t_i) \leq v_i(\widehat{x}_i)$. Hence $t_i$ is another maximizer of $v_i$ on $\varphi_i^n(p,q)$. We claim that $t_i \in \zeta_i^n(p,q)$. Indeed, let $z' \in X_i^n$ such that $v_i(z_i') > v_i(t_i)$. We have $p \cdot z' > p \cdot e_i + q$. Thus,

$$\forall \lambda \in \left]0,1\right], p \cdot ((1-\lambda)t_i + \lambda z') > p \cdot e_i + q.$$

Second we show that $\zeta_i^n(p,q)$ is convex valued.

Let $x$ and $x'$ be contained in $\zeta_i^n(p,q)$ and let $y \in \widetilde{P}_i^n(\lambda x + (1-\lambda)x')$ for $\lambda \in \left]0,1\right[$.

(a) First assume $p \cdot x < p \cdot e_i + q$ and $p \cdot x' \leq p \cdot e_i + q$. If $v_i(x) > v_i(x')$ then $p \cdot x \geq p \cdot e_i + q$, which is a contradiction. Hence $v_i(x) \leq v_i(x')$. If $v_i(x') > v_i(x)$, then $p \cdot x' = p \cdot e_i + q$. Because $v_i(x') > v_i(x)$, we have $\lambda x + (1-\lambda)x' \in \widetilde{P}_i^n(x)$ which implies that

$$p \cdot (\lambda x + (1-\lambda)x') \geq p \cdot e_i + q,$$

Thus, we have a contradiction because

$$p \cdot (\lambda x + (1-\lambda)x') < p \cdot e_i + q.$$

Therefore $v_i(x') = v_i(x)$. But now by quasi-concavity, we have

$$v_i(\lambda x + (1-\lambda)x') \geq v_i(x) = v_i(x').$$

If $v_i(\lambda x + (1-\lambda)x') > v_i(x)$, then

$$p \cdot (\lambda x + (1-\lambda)x') \geq p \cdot e_i + q,$$

a contradiction as before. Hence,

$$v_i(\lambda x + (1-\lambda)x') = v_i(x) = v_i(x').$$

Let $y \in \widetilde{P}_i^n(\lambda x + (1-\lambda)x')$, i.e., $y = \alpha(\lambda x + (1-\lambda)x') + (1-\alpha)z$ for some $\alpha \in \left[0,1\right[$, and some $z \in X_i^n$ such that $v_i(z) > v_i(\lambda x + (1-\lambda)x')$. We have the identity

$$\alpha(\lambda x + (1-\lambda)x') + (1-\alpha)z = \lambda(\alpha x + (1-\alpha)z) + (1-\lambda)(\alpha x' + (1-\alpha)z).$$

But we have, $p \cdot (\alpha x + (1-\alpha)z) \geq p \cdot e_i + q$, and $p \cdot (\alpha x' + (1-\alpha)z) \geq p \cdot e_i + q$. Therefore, $p \cdot y \geq p \cdot e_i + q$.

(b) Assume now $p \cdot x = p \cdot e_i + q$ and $p \cdot x' = p \cdot e_i + q$. In this case $p \cdot (\lambda x + (1-\lambda)x') = p \cdot e_i + q$. Let

$$y = \alpha(\lambda x + (1-\lambda)x') + (1-\alpha)z$$

for some $\alpha \in [0,1[$ and some $z \in X_i^n$ such that $v_i(z) > v_i(\lambda x + (1-\lambda)x')$. We have

$$v_i(z) > v_i(\lambda x + (1-\lambda)x') \geq \min\left\{v_i(x), v_i(x')\right\}.$$

Hence $p \cdot z \geq p \cdot e_i + q$, and $p \cdot y \geq p \cdot e_i + q$.

Finally, we show that $\zeta_i^n(\cdot,\cdot)$ has a closed graph. Let

$$x_i^\nu \in \zeta_i^n(p^\nu, q^\nu),\ x_i^\nu \to x,\ (p^\nu, q^\nu) \to (p,q),$$

and let

$$z = (1-\lambda)x_i + \lambda y,$$

for $\lambda \in ]0,1]$ and $y \in X_i^n$ such that $v_i(y) > v_i(x)$. By the u.s.c. of $v_i$, for $\nu$ large enough, $v_i(y) > v_i(x_i^\nu)$. Let

$$z^\nu = (1-\lambda)x_i^\nu + \lambda y.$$

Clearly, $z^\nu \in \widetilde{P}_i^n(x_i^\nu)$, so that

$$p^\nu \cdot z^\nu \geq p^\nu \cdot e_i + q^\nu.$$

Since $\lim_{\nu \to +\infty} z^\nu = z$,

$$p \cdot z \geq p \cdot e_i + q.$$

Thus, $x \in \zeta_i^n(p,q)$.□

Now, define

$$Z^n(p,q) := [\sum_{i=1}^m (\zeta_i^n(p,q) - e_i^\perp)] \times \{-m\}.$$

It is clear that,

$$\forall (p,q) \in (D \times \mathbb{R}_+) \cap \Pi, \forall x \in Z^n(p,q), (p,q).x \leq 0.$$

We can now apply the Debreu fixed point lemma (see Florenzano and Le Van (1986)).

**Lemma 2.** *Let $P \subset \mathbb{R}^{\ell+1}$ be a convex cone which is not a linear subspace. Let $P^0$ and $\Pi$ denote respectively the polar of $P$ and the unit sphere of $\mathbb{R}^{\ell+1}$. Let $Z$ be an upper semicontinuous (u.s.c.), nonempty, compact and convex valued correspondence from $P \cap \Pi$ into $\mathbb{R}^{\ell+1}$ such that*

$$\forall p \in P \cap \Pi, \exists z \in Z(p) \text{ such that } p \cdot z \leq 0.$$

*Then there exists $\bar{p} \in P \cap \Pi$ such that $Z(\bar{p}) \cap P^0 \neq \emptyset$.*

Thus, it follows from the above lemma that

$$\begin{gathered} \exists (p^n, q^n) \in (D \times \mathbb{R}_+) \cap \Pi, \\ \exists x_i^n \in \zeta_i(p^n, q^n), \forall i, \\ \text{and} \\ \exists z^n \in \textstyle\sum_{i=1}^m L_i \text{ such that } \sum_{i=1}^m (x_i^n - e_i^\perp) = z^n. \end{gathered}$$

One can write $z^n = \sum_{i=1}^m l_i^n$, where $l_i^n \in L_i, \forall i$. Then one has

$$\sum_{i=1}^m (x_i^n - l_i^n + \hat{e}_i) = \sum_{i=1}^m e_i,$$

where $e_i = e_i^\perp + \hat{e}_i$. Therefore $(x_i^n) \in \mathcal{A}^\perp$ where $\mathcal{A}^\perp$ is the projection of $\mathcal{A}$ onto $\prod_{i=1}^m L_i^\perp$. From Theorem 1 in Allouch, Le Van, and Page (2002), Hart's condition holds if and only if the set $\mathcal{A}^\perp$ is compact. Passing to a subsequence if necessary, it follows from the compactness of $\mathcal{A}^\perp$ and $(D \times \mathbb{R}_+) \cap \Pi$ that $\lim_{n \to +\infty} (x_i^n) = x^* \in \mathcal{A}^\perp$ and

$$\lim_{n \to +\infty} (p^n, q^n) = (p^*, q^*) \in (D \times \mathbb{R}_+) \cap \Pi.$$

Since $x^* \in \mathcal{A}^\perp$ there exists $(l_i) \in \prod_{i=1}^m L_i$ such that

$$\sum_{i=1}^m (x_i^* + l_i) = \sum_{i=1}^m e_i.$$

Letting $x_i'^* = x_i^* + l_i$, we have by Global Nonsatiation of $v_i$ that there exists $z_i \in X_i$, such that

$$v_i(z_i) > v_i(x_i'^*) = v_i(x_i^* + l_i) = v_i(x_i^*).$$

Then, by weak uniformity, [A.3], there exists $z_i^\perp \in X_i \cap L_i^\perp$ such that $v_i(z_i^\perp) > v_i(x_i^*)$. For $n$ large enough, $z_i^\perp \in X_i^n$, and therefore $v_i(z_i^\perp) > v_i(x_i^n)$ (since $v_i$ is u.s.c.). It follows from $x_i^n \in \zeta_i^n(p^n, q^n)$, that

$$p^n \cdot y_i^n \geq p^n \cdot e_i + q^n, \text{ for } y_i^n = (1 - \lambda) x_i^n + \lambda z_i^\perp, \ \lambda \in \left]0, 1\right].$$

Thus,

$$\begin{aligned}
&p^n \cdot y_i^n = p^n \cdot \left((1-\lambda)x_i^n + \lambda z_i^{\perp}\right)\\
&= p^n \cdot ((1-\lambda)x_i^n + \lambda z_i)\\
&\geq p^n \cdot e_i + q^n, \text{ for } \lambda \in \left]0,1\right],
\end{aligned}$$

Letting $n \to \infty$ we obtain

$$p^* \cdot ((1-\lambda)x_i^* + \lambda z_i) \geq p^* \cdot e_i + q^*.$$

Next letting $\lambda \to 0$ we obtain

$$p^* \cdot x_i^* \geq p^* \cdot e_i + q^*.$$

But, $p^* \cdot x_i^* \leq p^* \cdot e_i + q^*$. Hence

$$p^* \cdot x_i^* = p^* \cdot e_i + q^*, \forall i,$$

Summing over $i$, we obtain $q^* = 0$, and $p^* \cdot x_i^* = p^* \cdot e_i, \forall i$.

We claim that $(x_i'^*, p^*)$ is a quasi-equilibrium of $\mathcal{E}'$. Thus, it remains to check that $v_i(x_i) > v_i(x_i'^*)$ implies $p^* \cdot x_i \geq p^* \cdot e_i$. For such an $x_i$, write $x_i = x_i^{\perp} + \widetilde{l}_i$ for $x_i^{\perp}$ the projection of $x_i$ on $L_i^{\perp}$ and $\widetilde{l}_i \in L_i$. We have

$$v_i(x_i) = v_i(x_i^{\perp} + \widetilde{l}_i) = v_i(x_i^{\perp}) > v_i(x_i'^*) = v_i(x_i^* + l_i) = v_i(x_i^*).$$

For $n$ large enough, $x_i^{\perp} \in X_i^n$. Since $v_i(x_i^{\perp}) > v_i(x_i^*)$ and $\lim_{n\to+\infty}(x_i^n) = x^*$, the upper semicontinuity of $v_i$ implies that for $n$ large enough $v_i(x_i^{\perp}) > v_i(x_i^n)$. Since $x_i^n \in \zeta_i^n(p^n, q^n)$,

$$p^n \cdot x_i^{\perp} \geq p^n \cdot e_i + q^n,$$

which implies $p^* \cdot x_i^{\perp} \geq p^* \cdot e_i$. Finally, since $p^* \cdot x_i^{\perp} = p^* \cdot (x_i^{\perp} + \widetilde{l}_i) = p^* \cdot x_i$, we have $p^* \cdot x_i \geq p^* \cdot e_i$. ■

## References


[1] Allouch N., Le Van, C., Page, F.H.Jr., The geometry of arbitrage and the existence of competitive equilibrium, Journal of Mathematical Economics **38**, 373-392, (2002).

[2] Allouch N., Florenzano, M., Edgeworth and Walrasian equilibrium of an arbitrage-free exchange economy, Economic Theory **23**, 353-371, (2004).

[3] Bergstrom, T. C., How to discard 'free disposability' - at no cost, Journal of Mathematical Economics **3**, 131-134, (1976).

[4] Dana, R.-A., Le Van, C., Magnien F., On the different notions of arbitrage and existence of equilibrium, Journal of Economic Theory **86**, 169-193, (1999).

[5] Debreu, G., Theory of Value, John Wiley and Sons, New-York, (1959).

[6] Florenzano, M., Le Van, C., A note on Gale-Nikaido-Debreu lemma and the existence of general equilibrium, Economics Letters **32**, 107-110, (1986).

[7] Gale, D., Mas-Colell, A., An equilibrium existence theorem for a general model without ordered preferences, Journal of Mathematical Economics **2**, 9-15, (1975).

[8] Geistdoerfer-Florenzano, M., The Gale-Nikaido-Debreu lemma and the existence of transitive equilibrium with or without the free-disposal assumption, Journal of Mathematical Economics **9**, 113-134 (1982).

[9] Grandmont, J.M., On the Temporary Competitive Equilibrium, Working Paper, No. 305, Center for Research in Management Science, University of California, Berkeley, August (1970).

[10] Grandmont, J.M., Continuity Properties of a von Neumann-Morgenstern Utility, Journal of Economic Theory **5**, 45-57, (1972).

[11] Grandmont, J.M., Temporary general equilibrium theory, Econometrica **45**, 535-572, (1977).

[12] Green, J.R., Temporary general equilibrium in a sequential trading model with spot and futures transactions, Econometrica **41**, 1103-1124, (1973).

[13] Hammond, P.J., Overlapping expectations and Hart's condition for equilibrium in a securities model, Journal of Economics Theory **31**, 170-175, (1983).

[14] Hart, O., On the existence of an equilibrium in a securities model, Journal of Economic Theory **9**, 293-311, (1974).

[15] Milne, F., Short selling, default risk and the existence of equilibrium in a securities model, International Economic Review **21**, 255-267, (1980).

[16] Nielsen, L.T., Asset market equilibrium with short selling, Review of Economic Studies **56**, 467-474, (1989)

[17] PAGE, F.H.JR., On equilibrium in Hart's securities exchange model, Journal of Economic Theory **41**, 392-404, (1987).

[18] PAGE, F.H.JR., WOODERS, M.H., A necessary and sufficient condition for compactness of individually rational and feasible outcomes and the existence of an equilibrium, Economics Letters **52**, 153-162, (1996).

[19] PAGE, F.H.JR., WOODERS, M.H., MONTEIRO, P.K., Inconsequential arbitrage, Journal of Mathematical Economics **34**, 439-469, (2000).

[20] ROCKAFELLAR, R.T., Convex Analysis, Princeton University Press, (1970).

[21] WERNER J., Arbitrage and the existence of competitive equilibrium, Econometrica **55**, 1403-1418 (1987).